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LOWER AND UPPER ESTIMATES OF THE RUPTURE TIME FOR STRUCTURAL ELEMENTS

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A method of obtaining the lower and upper estimates of the rupture time for structural elements is proposed. As an example, the rupture time for a rotating, nonuniformly heated disk with a hyperbolic profile is estimated.

In designing the structural elements that indented for work under creep conditions, it is necessary that the elements satisfy both the strength and rigidity criteria during the period of their use. To ensure these criteria, the rupture and service times, respectively, should be calculated [1–4].

Using Rabotnov's kinetic theory of creep, one can calculate the lower and upper estimates for the rupture time of structures. Below, we obtain these estimates for an arbitrary nonuniformly heated body (structural element) subjected to constant-in-time surface loads. The temperature field is assumed to be stationary. The estimates obtained are compared with those currently used in designing the structures [2, 5].

1. Relations of Rabotnov's kinetic theory of creep [1, 2, 6]

$$\eta_{ij} = \frac{W}{\sigma_{\text{eff}}} \frac{\partial \sigma_{\text{eff}}}{\partial \sigma_{ij}}, \qquad W = \frac{B\sigma_{\text{eff}}^{n+1}}{\varphi(\omega)}, \qquad i,\, j = 1,\, 2,\, 3; \tag{1.1}$$

$$\dot{\omega} = B_1 \sigma_*^{g+1}/\varphi(\omega), \qquad \omega(x_k, 0) = 0, \qquad \omega(x_k^*, t_*) = 1 \tag{1.2}$$

and the equations of equilibrium, the Cauchy relations, the strain-rate compatibility equations, and the corresponding boundary conditions make it possible to combine two independent problems: the problem of determining the stress–strain state of an arbitrary body and the problem of calculating the rupture time of this body [1]. Obviously, its solution involves certain mathematical difficulties even in the case of the simplest structural elements [1]. Nikitenko and Zaev [7, 8] developed an approximate method of solving this problem, according to which the stress–strain state of an arbitrary, nonuniformly heated body under constant surface loads is represented in the form (the temperature field is assumed to be constant)

$$\sigma_{ij}(x_k, t) = \sigma_{ij}^0 f(x_k, t) + C_1(x_k, t)\delta_{ij}, \qquad f = [\varphi(\omega)]^{1/n}/X(t); \tag{1.3}$$

$$\dot{\epsilon}_{ij}(x_k, t) = \eta_{ij}^0 F(t). \tag{1.4}$$

In (1.1)–(1.4), σ_{eff} and σ_* are first-degree homogeneous functions of stresses, $W = \sigma_{ij}\eta_{ij}$, σ_{ij} and η_{ij} are the components of the creep-stress and strain-rate tensors, B, n, B_1, and g are the creep and long-term strength characteristics of the material (n and g are assumed to be constant in the temperature range considered and B and B_1 are assumed to be temperature functions [9]), and ω is the parameter governing the accumulation of defects in the material. For the initial material, the damage parameter is equal to zero at each point of the body with the coordinates x_k ($k = 1,\, 2,\, 3$); if $\omega(x_k^*, t_*) = 1$ at a certain point with the coordinates

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x_k^* at the moment $t = t_*$, it is assumed that the rupture begins at this point and the time t_* is called the rupture time. The function $\varphi(\omega)$ in relations (1.1) and (1.2) is taken in the form $\varphi = \omega^\alpha(1-\omega^{\alpha+1})^m$ [6]; the case $m = 0$, $\alpha \neq 0$ refers to the material that hardens in creep, the case $m \neq 0$m $\alpha = 0$ refers to the material that unhardens in creep, and the case $\alpha = m = 0$ refers to the material that obeys the law of steady creep; $\epsilon_{ij} = \varepsilon_{ij} + p_{ij} + \varepsilon\delta_{ij}$ are the strain-tensor components, ε_{ij} are the components of the elastic-strain tensor, $\varepsilon\delta_{ij}$ is the temperature strain, and p_{ij} are the creep strains. The superscript 0 shows that the function depends only on the coordinates x_k and the dot denotes differentiation with respect to time.

Generally, σ_{ij}^0 and η_{ij}^0 in (1.3) and (1.4) are the statically admissible and kinematically possible fields. In this case, this is the solution of the above-formulated problem under the assumption of steady creep of the material

$$\eta_{ij}^0 = \frac{W^0}{\sigma_{\text{eff}}^0}\frac{\partial\sigma_{\text{eff}}^0}{\partial\sigma_{ij}^0}, \qquad W^0 = B^0(\sigma_{\text{eff}}^0)^{n+1}. \tag{1.5}$$

The hydrostatic component C_1 satisfies the system

$$\frac{\partial C_1}{\partial x_j}\delta_{ij} = -\sigma_{ij}^0\frac{\partial f}{\partial x_j}$$

at each point of the body and the condition $C_1 = (1-f)T$ at the body surface S_T, where T is the modulus of surface-load vector.

The functions $F(t)$, $X(t)$, and $\omega(x_k, t)$ are determined from a system which has the following form for the material unhardening in creep ($\alpha = 0$ and $m \neq 0$) [7, 8]:

$$F(t) = \int\limits_V f(x_k,t)\sigma_{ij}^0\dot{\varepsilon}_{ij}\,dV \Big/ \int\limits_V W^0\,dV + [X(t)]^{-n};$$

$$\int\limits_1^\mu Z^{m(n-g-1)/n}\,dZ = -[(m+1)t^0]^{-1}\int\limits_0^t [X(\tau)]^{-(g+1)}\,d\tau; \tag{1.6}$$

$$\int\limits_V W^0[\mu(x_k,t)]^{m/n}\,dV = X(t)\int\limits_V W^0\,dV. \tag{1.7}$$

Here $\mu(x_k,t) = 1-\omega(x_k,t)$, $\mu(x_k,0) = 1$, $\mu(x_k^*,t_*) = 0$, and

$$t^0(x_k) = [CB_1^0(\sigma_*^0)^{g+1}]^{-1}, \qquad C^{-1} = \int\limits_0^1 \varphi(\omega)\,d\omega. \tag{1.8}$$

We note that (1.8) follows from (1.2) under the assumption of steady creep of the material. From (1.8), the rupture time $t^0(x_k^*)$ at the point with the coordinates x_k^* is calculated. Hereafter, we denote it by t_*^0. Thus, t_*^0 is the moment at which the body begins to fail at the point with the coordinates x_k^* in solving a corresponding problem under the assumption that the material obeys the law of steady creep (1.5).

The requirement for the convergence of the integral on the left side of Eq. (1.6) imposes certain restrictions on the creep and long-term strength characteristics of the material. These restrictions do not contradict experimental results. In particular, $\beta = m/[n + m(n-g-1)] > 0$. One can verify that system (1.6), (1.7) admits an analytical solution only for $\beta = 1$; in other cases, one can obtain the lower and upper estimates for this solution. For example, for $\beta > 1$, we have

$$\Big(1-\frac{t}{\bar{t}^0}\Big)^{\beta v} \leqslant X(t) \leqslant \Big(1-\frac{v(g+2)t}{\bar{t}^0}\Big)^{1/(g+2)},$$

$$\mu^{m/n} \geqslant \Big\{1-\frac{\bar{t}^0}{t_*^0}\Big[1-\Big(1-\frac{t}{\bar{t}^0}\Big)^v\Big]\Big\}^\beta, \tag{1.9}$$

$$\mu^{m/n} \leqslant \left\{1 - \frac{\bar t^0}{t^0_*}\left[1 - \left(1 - \frac{v(g+2)t}{\bar t^0}\right)^{1/(g+2)}\right]\right\}^{\beta}. \tag{1.10}$$

Here

$$v = \frac{n + m(n-g-1)}{n(m+1)}, \qquad \bar t^0 = \int_V W^0\,dV \Big/ \int_V (W^0/t^0)\,dV. \tag{1.11}$$

With allowance for $0 < \mu(x_k, t) < 1$ and $\mu(x_k^*, t_*) = 0$, from (1.9) and (1.10) we obtain the desired estimates of the rupture time

$$\frac{1-(1-\lambda)^{1/v}}{\lambda} \leqslant \frac{t_*}{t^0_*} \leqslant \frac{1-(1-\lambda)^{g+2}}{\lambda}\,\frac{1}{v(g+2)}, \qquad \lambda = \frac{t^0_*}{\bar t^0}. \tag{1.12}$$

2. In calculating the structures, one is recommended to calculate the lower estimate for the rupture time under the assumption of steady creep of the material [1, 2, 5, 10]. It is impossible to show that such a calculation gives the lower estimate [10]. Similarly, without substantiation, one is recommended to calculate the upper estimate by analyzing the limit state of the body [2]. Thus, these estimates can be written in the form

$$t^0_* \leqslant t_* \leqslant t_{**}. \tag{2.1}$$

Here t_{**} is the moment the failure occurs at each point of the body.

We now compare the estimates (1.12) and (2.1). We first perform a limit-state calculation. We require that the stresses satisfy the limit-state condition [11]. This condition follows from (1.2) and, for the stationary thermal–force action, it has the form [11]

$$B^0_1(\sigma^0_*)^{g+1} = (Ct_{**})^{-1}, \qquad C^{-1} = \int_0^1 \varphi(\omega)\,d\omega. \tag{2.2}$$

Relation (2.2) can be written in the form

$$U\sigma^0_* = \sigma_{\text{l.s.}}, \qquad U = [B^0_1(\theta)/B^0_1(\theta_0)]^{1/(g+1)}, \qquad \sigma_{\text{l.s.}} = [CB^0_1(\theta_0)t_{**}]^{-1/(g+1)},$$

where $\sigma_{\text{l.s.}}$ is the ultimate long-term tensile strength of the material determined for t_{**} at a fixed temperature θ_0. The limit-state condition can be written as follows: $\sigma^0_* = \sigma_{\text{l.s.}}$, where $\sigma_{\text{l.s.}} = [CB^0_1(\theta)t_{**}]^{-1/(g+1)}$. This condition is an analog of the yield condition for the material. In this case, the ultimate long-term strength of the material determined for t_{**} is a known temperature function.

Using condition (2.2), from (1.8) and (1.11) we obtain $t^0_* = t_{**}$ and $\bar t^0 = t_{**}$. Taking into account that $\lambda = t^0_*/\bar t^0 = 1$, from (1.12) we finally obtain

$$t_{**} \leqslant t_* \leqslant t_{**}/[v(g+2)]. \tag{2.3}$$

Comparing (2.1) and (2.3) with allowance for $t^0_* = t_{**}$, we infer that the upper estimate (2.1) does not agree with (2.3).

In most practical cases, the limit-state condition (2.2) cannot be satisfied. Therefore, as a rule,

$$B^0_1(\sigma^0_*)^{g+1} \leqslant (Ct_{**})^{-1}. \tag{2.4}$$

With allowance for (2.4), (1.8) implies that $t^0_* \geqslant t_{**}$. One can verify that the fraction in (1.12) is no less than unity: $(1-(1-\lambda)^{1/v})/\lambda \geqslant 1$, $(1-(1-\lambda)^{g+2})/\lambda \geqslant 1$, and $0 < \lambda \leqslant 1$. By virtue of these inequalities and (1.12), we obtain

$$t_* \leqslant \frac{1-(1-\lambda)^{g+2}}{\lambda}\,\frac{t^0_*}{v(g+2)};$$

$$\frac{1-(1-\lambda)^{g+2}}{\lambda}\,\frac{t^0_*}{v(g+2)} \geqslant \frac{t^0_*}{v(g+2)} \geqslant \frac{t_{**}}{v(g+2)}; \tag{2.5}$$

$$t_* \geqslant \frac{1-(1-\lambda)^{1/v}}{\lambda} t_*^0 \geqslant t_*^0 \geqslant t_{**}. \qquad (2.6)$$

System (2.6) shows the consistency of the lower estimate of the rupture time (2.1) used in calculational practice. The rupture time calculated on the basis of the estimate (1.12) is closer to the true value than that calculated with the use of the estimate (2.1).

3. Let us estimate the time the rupture begins in a rotating, nonuniformly heated disk with a hyperbolic profile. In this case, the plane stress state occurs. We assume that $\sigma_\varphi > \sigma_r > \sigma_z = 0$.

As an equivalent stress σ_{eff}^0 in (1.5), we use the maximum shear-stress criterion $\sigma_{\text{eff}}^0 = \sigma_\varphi^0/2$. We assume that the creep coefficient is a power function of temperature: $B^0 = B_0\theta^{\nu_1}$. The law of temperature variation on the disk radius is taken in the form

$$\theta = \theta_0 (r/a)^{\nu_2} \qquad (\nu_2 \geqslant 0),$$

where a is the inner radius of the disk. Let $\varrho = b/a$, where b is the outer radius of the disk. The disk thickness is given by $h = h_0 r^{-k}$, where $k \geqslant 0$. The problem of determining the stress–strain state of a nonuniformly heated disk with a hyperbolic profile was solved in [9, 11] under the assumption of steady creep of the material for the power law (1.5).

The stresses are given by

$$\sigma_r^0 = C_1\Big(\frac{r}{a}\Big)^{k-1} - C_2\Big(\frac{a}{r}\Big)^{(1+\nu)/k} - \frac{\rho\Omega^2}{3-k} r^2, \qquad \sigma_\varphi^0 = \delta C_2\Big(\frac{a}{r}\Big)^{(1+\nu)/n},$$

$$C_1 = \frac{p_1 + p_2\varrho^{1+\delta-k}}{\varrho^\delta - 1} + \frac{\varrho^{3+\delta-k} - 1}{\varrho^\delta - 1}\,\frac{\rho a^2\Omega^2}{3-k}, \qquad \delta = k - 1 + \frac{1+\nu}{n}, \qquad (3.1)$$

$$C_2 = \frac{\big(p_1 + p_2\varrho^{1-k}\big)\varrho^\delta}{\varrho^\delta - 1} + \frac{(\varrho^{3-k} - 1)\varrho^\delta}{\varrho^\delta - 1}\,\frac{\rho a^2\Omega^2}{3-k},$$

where Ω is the angular velocity, ρ is the density of the disk material, $p_1 = -\sigma_r(a)$, $p_2 = \sigma_r(b)$, and $\nu = \nu_1\nu_2$.

As an equivalent stress σ_* in (2.2), we use the Johnson criterion $\sigma_* = \sigma_\varphi^0$. The coefficient B_1^0 is assumed to be a power temperature function: $B_1^0 = B_{01}\theta^{\nu_1}$. The limit-state condition (2.2) takes the form

$$(r/a)^{\nu/(g+1)}\sigma_\varphi^0 = \sigma_{\text{l.s.}}, \qquad \sigma_{\text{l.s.}} = (CB_{01}\theta_0^{\nu_1} t_{**})^{-1/(g+1)}. \qquad (3.2)$$

In the case of the limit state of the disk material, the stress field (3.1) must satisfy condition (3.2), which implies that the limit state of the disk occurs provided [11]

$$\nu = \frac{g+1}{n-g-1}, \qquad \frac{\rho a^2\Omega_*^2}{3-k} = \frac{\varrho^\delta - 1}{\delta\varrho^\delta(\varrho^{3-k} - 1)}\,\sigma_{\text{l.s.}} - \frac{p_1 + p_2\varrho^{1-k}}{\delta(\varrho^{3-k} - 1)}. \qquad (3.3)$$

We set $\nu = \mu(g+1)/(n-g-1)$ and $0 \leqslant \mu \leqslant 1$. The value of $\mu = 0$ corresponds to a uniformly heated disk, and $\mu = 1$ to a nonuniformly heated disk in the limit state.

The disk begins to fail at the inner surface: $r^* = a$. Using (3.1), we calculate t_*^0 and $\bar{t}^0$ from formulas (1.8) and (1.11) and their ratio $\lambda = t_*^0/\bar{t}^0$. We obtain

$$\lambda = \frac{\xi_2}{\xi_1}\,\frac{\varrho^{\xi_2} - 1}{\varrho^{\xi_1} - 1}, \qquad \lambda = \lambda(\mu),$$
$$\xi_1 = -k + \frac{n-g-2}{n}\,\frac{n-(1-\mu)(g+1)}{n-g-1}, \qquad \xi_2 = -k + 1 - \frac{n-(1-\mu)(g+1)}{n(n-g-1)}. \qquad (3.4)$$

Figure 1 shows curves which refer to the lower and upper bounds for the rupture time of the disk at various values of k ($k = 0$, 1, and 2) and $\varrho = 4$. The calculation was performed by formulas (1.12) and (3.4). The following characteristics of the creep and long-term strength of the material were used: $n = 6$, $g = 4.75$, and $m = 10$. The dashed curve refers to the upper bound, the solid curve to the lower bound, the dot-and-dashed curve to the lower bound calculated by (2.1), and the dotted curve to the upper bound

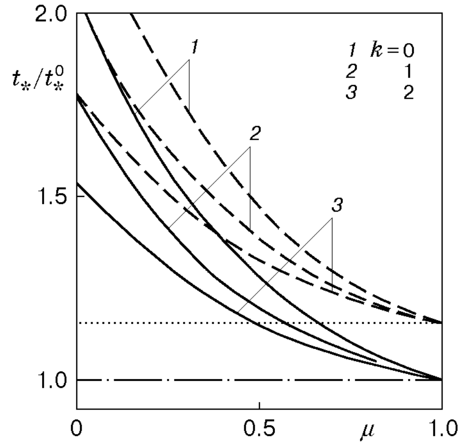

Fig. 1

calculated by (2.3). Obviously, the estimate calculated from (2.1) is close to (1.12) in the "neighborhood" of the value of μ corresponding to the limit state of the body. A state close to the limit state can occur in a body with a given geometry under external temperature-force actions and in a body with the optimal geometry under specified external actions. Indeed, the dependences shown in Fig. 1 imply that:

1) For a nonuniformly heated disk of constant thickness ($k = 0$), the dot-and-dashed curve deviates from the solid curve by 7, 19, and 63% for $\mu = 0.8$, 0.60, and 0.2, respectively;

2) For a nonuniformly heated disk with a hyperbolic profile ($k > 0$), these deviations are 5.5 ($k = 1$) and 4.5% ($k = 2$) for $\mu = 0.8$, 14 ($k = 1$) and 11% ($k = 2$) for $\mu = 0.6$, and 46 ($k = 1$) and 33% ($k = 2$), for $\mu = 0.2$;

3) For a uniformly heated disk ($\mu = 0$), these deviations are 104, 76, and 54% for $k = 0$, 1, and 2, respectively.

Thus, in all the cases, the lower estimate $t_* \geqslant t_*^0$ becomes rough for states different from the limit state.

In summary, we note that the corresponding estimates of the rupture time given by inequalities (1.12) are valid if $\beta = m/[n+m(n-g-1)] > 1$. If $\beta = 1$, the lower and upper estimates coincide, and the expression for calculating the rupture time becomes

$$\frac{t_*}{t_*^0} = \frac{1-(1-\lambda)^{1/\gamma}}{\lambda}, \qquad \gamma = \frac{m}{n(m+1)}$$

(if $\beta = 1$, then $v = \gamma$ and $v(g+2) = 1$ [7, 8]). The case $0 < \beta < 1$ is analyzed in a similar manner.

The above-considered method of calculating the lower and upper estimates for the rupture time of structural elements can be used for calculating the structures. In accordance with this method, one should determine the stresses in the body under the assumption that the creep is steady for the power law (1.5) and calculate the estimates by means of the system of inequalities (1.12) and relations (1.8) and (1.11).

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